Proofs, Set Theory, Natural Numbers, and Integers

Sept 2022

1. Use our definitions of integer even and odd parities to prove the following properties, given $x, y \in \mathbb{Z}$, where x is even and y is odd (assume exponential properties of integers are given):
   a. Prove $x + y$ has odd parity.
   b. Prove x^2 has even parity.
   c. Prove xy has even parity
   d. Prove y^x has odd parity (where $\mathrm{F} = \mathrm{N}$).
   e. Prove x^y has even parity (where $\mathrm{F} = \mathrm{N}$).
2. Prove the following statements about sets:
   a. $X \setminus A^c = A$, where $A \subset X$
   b. $|\emptyset| = 0$, by logic
   c. If $P(A) \subset P(B)$, then $A \subset B$
   d. $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$
   e. $(A \cup B)^c = A^c \cap B^c$
   f. Prove $\emptyset$ is unique
   h. Prove $A + A = A$ given our definition of Minkowski addition
3. Let $A \times B \subseteq B \times C$. Prove $A \subseteq C$. Which condition does this not hold?
4. Using the field axioms of real numbers, prove the following statements:
   a. If $x + y = x + z$ then $y = z$
   b. If $x + y = x$ then y = 0
   c. If $x + y = 0$ then $y = -x$
   d. $-(-x) = x$
5. *Carefully prove* the quadratic formula using the real field axioms.
6. Prove $\mathbb{N}$ is *not* a field.
7. What does it mean for a set to be ordered? Give three examples of an ordered set and list two properties you might encounter with ordered sets in general.
8. Prove that $\sqrt{3}$ cannot be represented by a rational number.
9. Represent the entirety of real numbers using set builder notation.
10. Take the repeated power set of the empty set. Which number set does this represent?